JRAHS 2006 TRIAL HSC - EXT I

Question 1. **Marks**

(a) Solve for x: $\dfrac{1}{x-2} \geq 2$. **3**

(b) Find: $\lim\limits_{h \to 0}\left(\dfrac{\cos 2h - 1}{h}\right)$. **2**

(c) The point P divides $A(-1, 5)$ and $B(3, -2)$ in the ratio $r:1$.

(i) Find the coordinates of P in terms of r. **2**

(ii) Find the value of r when the line $2x - 3y + 4 = 0$ intersects the interval AB. **2**

(d) Evaluate $\displaystyle\int_0^1 (x^2 + 1)^3 dx$. **3**

Question 2. **[START A NEW PAGE]**

(a) A plate is initially heated to $55^0\ C$, and it then cools to $41^0\ C$ in 10 minutes.
If the surrounding temperature, $S^0\ C$, is $22^0\ C$ and assuming Newton's Law of Cooling:

$$\frac{dT}{dt} = -k(T - S).$$

(i) Find the temperature of the plate 25 minutes from the start of cooling (to 1 decimal place). **3**

(ii) Find the time for the plate to cool to $25^0\ C$ (to 1 decimal place). **2**

(iii) Sketch the graph of the rate of temperature, $\dfrac{dT}{dt}$, versus the temperature T. **1**

(b) The displacement x metres of a particle after t seconds, is given by:

$$x = 5\sin 3t - 7\cos 3t.$$

(i) Show that the motion of the particle is SHM. **2**

(ii) Find the maximum displacement. **1**

(iii) Find the time when the particle first passes through the centre of motion (correct to 1 decimal place). **2**

(iv) Sketch the graph of the acceleration $\ddot{x}$ versus displacement x. **1**

Question 3. **[START A NEW PAGE]** **Marks**

(a) Differentiate $\cos^{-1}\left(-\frac{1}{x}\right)$ with respect to x. Answer in simplified form. **3**

(b) (i) On the same set of axes, sketch the graphs of $y=\sin^{-1}x$ and $y=\tan^{-1}x$. **2**

(ii) Given that: $\int_0^1 \sin^{-1}x\,dx=\frac{\pi}{2}-1$, find the area of the region bounded by **3**

$y=\sin^{-1}x$, $y=\tan^{-1}x$ and $x=1$.

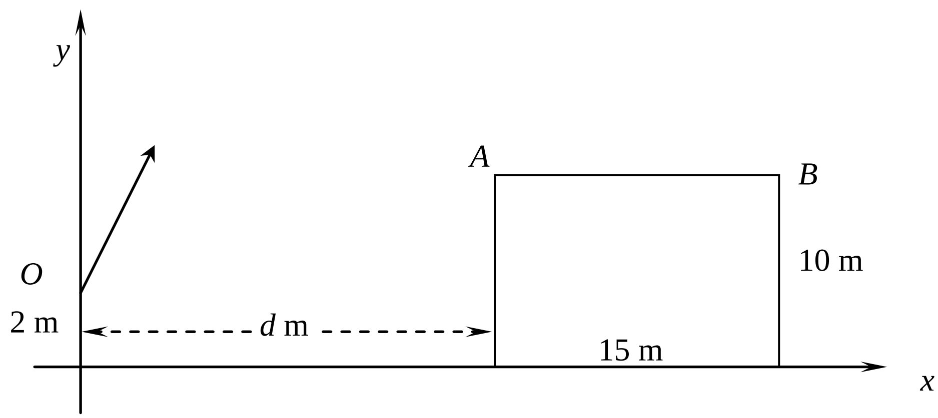
(c) (i) Show that $y=e^{-x}\sin 2x$ is a solution to the differential equation: **3**

$$y''+2y'+5y=0.$$

(ii) Hence, or otherwise, find $\int e^{-x}\sin 2x\,dx$. **1**

Question 4. **[START A NEW PAGE]**

(a) A fire truck arrives at a burning building 10 metres high and 15 metres wide. The water nozzle hose on the fire truck is 2 metres above the ground and d metres from the building, as shown in the diagram.



The angle of elevation of the hose, α, can be adjusted to range from 10^0 to 45^0. The parametric equations for the water particles from the nozzle are given by:

$x=30t\cos\alpha$ and $y=30t\sin\alpha-5t^2$, where t is the time in seconds when g = 10.

(i) Show that the trajectory path of the water is given by the equation: **1**

$$y=x\tan\alpha-\frac{x^2}{180}(1+\tan^2\alpha).$$

(ii) The hose nozzle is adjusted to an angle of elevation of 45^0. **2**
Find the distance, d, from the building if the water is to reach the furthest point B on top of the building as shown (answer to the nearest centimetre).

Q 4 *continues over the page*

Q 4 part (a) *continued* **Marks**

(iii) Find the angle of elevation α of the nozzle, for the water to reach position *A*, when the hose nozzle is 20 metres from the burning building (answer to nearest minute). **2**

(b) Find $\int \frac{4x-7}{2x^2+1}\,dx.$ **3**

(c) (i) For $t > 0,$ find the limiting sum of: $e^{-t} + e^{-2t} + e^{-3t} + \ldots.$ **1**

(ii) Hence, find an expression for the series; $e^{-t} + 2e^{-2t} + 3e^{-3t} + \ldots.$ **1**

(d) A semi-circle of radius *r* has the equation: $y = \sqrt{r^2 - x^2}.$

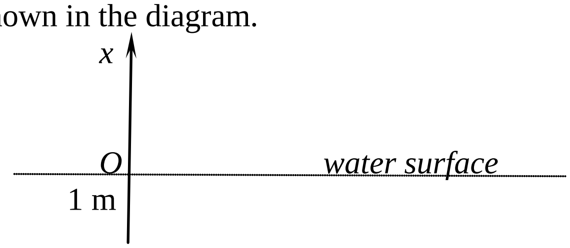
(i) Find $\frac{dy}{dx}$ at the point $P(x, y).$ **1**

(ii) Prove that the tangent, at any point *P* on the semi-circle, is perpendicular to the radius. **1**

Question 5. **[START A NEW PAGE]**

(a) Find the greatest coefficient in the expansion of $(4x+5)^{11}.$ **3**
(Leave the answer in index form).

(b) A ping pong ball is initially placed 1 metre beneath the surface of the water, as shown in the diagram.



The ping pong ball is released in the water with an acceleration of $\ddot{x}$ m/s^2, where $\ddot{x} = -625x,$ and where *x* metres is the displacement of the motion measured from the water surface.

(i) Is the motion of the ping pong ball only SHM? Give reasons. **1**

(ii) Prove that: $\frac{d}{dx}\left(\frac{v^2}{2}\right) = \ddot{x}.$ **2**

(iii) Find the expression for the ping pong ball's velocity *v* m/s when it is in the water. **2**

(iv) Find the velocity of the ball at the water's surface. **1**

(v) Assuming there is no air resistance and the acceleration due to gravity is 10 m/s^2, derive an expression for the displacement in air in terms of *v* **2**

(vi) Find the maximum height that the ping pong ball reaches above the surface of the water. **1**

Question 6. **[START A NEW PAGE]** **Marks**

(a) How many groups of 2 men and 2 women can be hosen from 6 men and 8 women? **2**

(b) Six letter words are formed from the letters of the word ***CYCLIC***.

(i) How many different 6-letter words can be formed? **2**

(ii) How many 6 letter words can be formed, if no '***C***'s are together? **2**

(iii) What is the probability of all the '***C***'s together, if it is known a vowel is at the end? **2**

(c) Prove, by the method of mathematical induction that: **4**

$$\sin q + \sin 3q + \sin 5q + \ldots + \sin(2n-1)q = \frac{1-\cos 2nq}{2\sin q}, \text{ for } n = 1, 2, 3, \ldots.$$

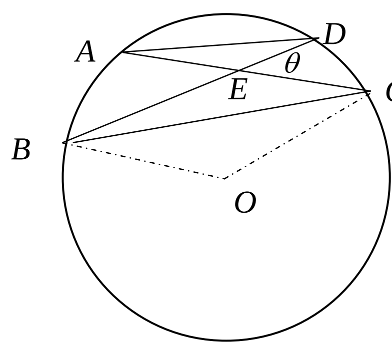
Question 7. **[START A NEW PAGE]**

(a) At the end of each month, for 15 years, a man invests \$400 at an interest rate Which is paid monthly at 6% *pa*.

(i) Show that the value of his first payment, at the end of 15 years, is \$976.75 **2**

(ii) Find the value of the man's total investment at the end of the 15 years. **2**

(b) A circle, centre O with a constant radius r, is such that the chords AC and BD intersect at point E, $\angle CED = \theta$ radians and $\angle BOC = \frac{2\pi}{3}$ radians, as shown the diagram.



Not to scale

(i) Show that the sum of the arcs AB and CD equal $2r\theta$, give reasons. **3**

(ii) Show that the perimeter P of the shape $ABCD$, where BC, AD are chords and CD, AB are arc lengths, is given by: **2**

$$P = r\left(2\theta + \sqrt{3} + 2\sin\left(\frac{\pi}{3} - \theta\right)\right).$$

(iii) Find the value of θ, in the domain $0 \le \theta \le \frac{\pi}{2}$ for the perimeter of $ABCD$ to have a maximum value. Justify your answer. **3**

2006 Mathematics Extension I Trial Exam. **1**

(a) $\frac{1}{x-2} \geq 2$ $\quad x \neq 2$

$(x-2) \geq 2(x-2)^2$

$(x-2) - 2(x-2)^2 \geq 0$

$(x-2)(1-2(x-2)) \geq 0$

$(x-2)(5-2x) \geq 0$

∴ Solution $2 < x \leq 2\frac{1}{2}$ (3)

(b) $\lim_{h\to 0}\left(\frac{\cos 3h-1}{h}\right)\frac{(\cos 3h+1)}{(\cos 3h+1)}$

$= \lim_{h\to 0}\frac{\cos^2 3h - 1}{h[\cos 3h+1]}$

$= \lim_{h\to 0}\frac{-\sin^2 3h}{h[\cos 3h+1]}$

$= \lim_{h\to 0}\frac{\sin 3h}{3h}\cdot\frac{-3\sin 3h}{\cos 3h+1}$

$= 1\cdot\frac{0}{2}$

$= 0$ (2)

(c) $r:1$

(i) $A(-1,5)$ $\quad B(3,-2)$

$P \equiv \left[\frac{3r-1}{r+1}, \frac{-2r+5}{r+1}\right]$ (2)

(ii) ∴ $2\left[\frac{3r-1}{r+1}\right] - 3\left[\frac{-2r+5}{r+1}\right] + 4 = 0$

$6r - 2 + 6r - 15 + 4r + 4 = 0$

$16r = 13$

$r = \frac{13}{16}$ (2)

(d) $\int_0^1 (x^2+1)^3\,dx$

$= \int_0^1 (x^6 + 3x^4 + 3x^2 + 1)\,dx$

$= \left[\frac{x^7}{7} + \frac{3}{5}x^5 + x^3 + x\right]_0^1$

$= 2\frac{26}{35}$ (3)

2 $\quad \frac{dT}{dt} = -k(T-T_0)$

$T = T_0 + Ae^{-kt}$

$T_0 = 22°$ And $t=0$ $T=55$

∴ $55 = 22 + Ae^0$

$A = 33 \Rightarrow T = 22 + 33e^{-kt}$

And $41 = 22 + 33e^{-10k}$

$e^{-10k} = \frac{19}{33}$

$k = \frac{1}{10}\ln\frac{33}{19}$

∴ $T = 22 + 33e^{-\frac{t}{10}\ln\left(\frac{33}{19}\right)}$ (2)

(i) $t = 25$

$T = 22 + 33e^{-\frac{25}{10}\ln\frac{33}{19}}$

$= 30.3°C$ (1)

(ii) $25 = 22 + 33e^{-\frac{t}{10}\ln\frac{33}{19}}$

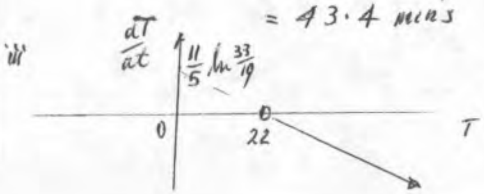
$e^{-\frac{t}{10}\ln\frac{33}{19}} = \frac{3}{33}$

$-\frac{t}{10}\ln\frac{33}{19} = -\ln 11$

$t = \frac{10\ln 11}{\ln\frac{33}{19}}$

$= 43.4$ mins (2)

(iii)



(1)

2

2(b) $x = 5\sin 3t - 7\cos 3t$

(i) $\dot{x} = 15\cos 3t + 21\sin 3t$

$\ddot{x} = -45\sin 3t + 63\cos 3t$

$= -9[5\sin 3t - 7\cos 3t]$

$\ddot{x} = -9x$ (1)

which is of the form $\ddot{x} = -n^2(x-b)$

$n = 3$ $\quad b = 0$

∴ motion SHM.

(ii) Max displacement $= \sqrt{5^2+7^2}$

$= \sqrt{25+49}$

$= \sqrt{74}$ units. (1)

Max velocity $= \sqrt{15^2+21^2}$

$= 3\sqrt{74}$ m/s (1)

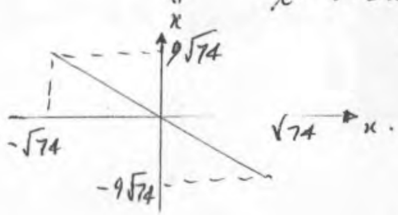
(iii) $x = 0$ $\quad 5\sin 3t - 7\cos 3t = 0$

$\tan 3t = 7/5$

$3t = \tan^{-1}\frac{7}{5}$

$t = 0.32$ s (2)

(iv)



(1)

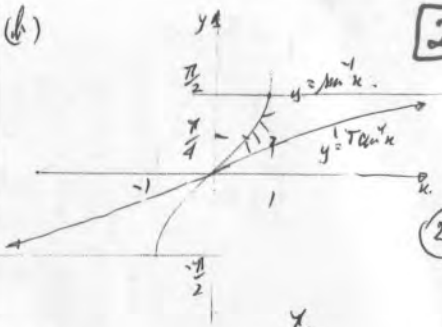
3(a) $\frac{d}{dx}\cos^{-1}\left(\frac{-1}{x}\right) = \frac{1}{x^2}\cdot\frac{-1}{\sqrt{1-\frac{1}{x^2}}}$

$= \frac{-\sqrt{x^2}}{x^2\sqrt{x^2-1}}$

$= \frac{-|x|}{x^2\sqrt{x^2-1}}$

$= \frac{-1}{|x|\sqrt{x^2-1}}$ (3)

(b)



(2)

$\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \int_0^{\pi/4}\tan y\,dy$

$= \frac{\pi}{4} - \int_0^{\pi/4}\frac{\sin y}{\cos y}\,dy$

$= \frac{\pi}{4} + \left[\ln\cos y\right]_0^{\pi/4}$

$= \frac{\pi}{4} + \ln\left(\frac{1/\sqrt{2}}{1}\right)$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$ (2)

∴ Region $\int_0^1 \sin^{-1}x\,dx \quad \int_0^1 \tan^{-1}x\,dx$

$= \frac{\pi}{2} - 1 - \left[\frac{\pi}{4} - \frac{1}{2}\ln 2\right]$

$= \frac{\pi}{4} - 1 + \frac{1}{2}\ln 2$ (1)

(c)(i) $y = e^{-x}\sin 2x$

$\frac{dy}{dx} = -e^{-x}\sin 2x + 2e^{-x}\cos 2x$

$= e^{-x}[2\cos 2x - \sin 2x]$ (1)

$\frac{d^2y}{dx^2} = -e^{-x}[2\cos 2x - \sin 2x] + e^{-x}[-4\sin 2x - 2\cos 2x]$ (1)

$= e^{-x}[-3\sin 2x - 4\cos 2x]$

∴ $y'' + 2y' + 5y = e^{-x}[-3\sin 2x - 4\cos 2x + 4\cos 2x - 2\sin 2x + 5\sin 2x]$

$= 0$ (1)

i) (c) $y'' + 2y' + 5y = 0$

$$y = -\frac{1}{5}\left[y'' + 2y'\right]$$

$$\int y\,dx = -\frac{1}{5}\int\left(y'' + 2y'\right)dx$$

$$= -\frac{1}{5}\left[y' + 2y\right] + C$$

$$\int e^{-x}\sin 2x\,dx = -\frac{1}{5}\left[e^{-x}\left(2\cos 2x - \sin 2x\right) + 2e^{-x}\sin 2x\right] + C$$

$$= \frac{-e^{-x}}{5}\left[2\cos 2x + \sin 2x\right] + C \qquad (1)$$

4(a)(i) $x = 30t\cos\alpha$ $\qquad y = -5t^2 + 30t\sin\alpha$

$t = \dfrac{x}{30\cos\alpha}$ $\qquad y = -5\left(\dfrac{x}{30\cos\alpha}\right)^2 + 30\sin\alpha\,\dfrac{x}{30\cos\alpha}$

$$y = \frac{-x^2}{180}\sec^2\alpha + x\tan\alpha$$

or $$y = \frac{-x^2}{180}\left[1 + \tan^2\alpha\right] + x\tan\alpha \qquad (1)$$

ii) $\alpha = 45^\circ$ $\qquad y = 8$ $\qquad x = d + 15$

$$\therefore\ 8 = \frac{-x^2}{180}\cdot(1+1) + x$$

$$x^2 - 90x + 720 = 0$$

$$x = \frac{90 \pm \sqrt{90^2 - 4\times 720}}{2}$$

$d + 15 = 81.12$ or 17.75.

$d = 66.12\,m$ furthest distance. $\qquad (2)$

iii) $A(20, 8)$

$$8 = -\frac{400}{180}\left[1 + \tan^2\alpha\right] + 20\tan\alpha$$

$$72 = -20\left[1 + \tan^2\alpha\right] + 180\tan\alpha$$

$$20\tan^2\alpha - 180\tan\alpha + 92 = 0$$

$$5\tan^2\alpha - 45\tan\alpha + 23 = 0$$

$$\tan\alpha = \frac{45 \pm \sqrt{45^2 - 4\times 5\times 23}}{2\times 5}$$

$$= 0.544 \text{ or } 8.46$$

Angle elevation $\alpha = 28^\circ 33'$ as $0 \le \alpha \le 45^\circ$. $\qquad (2)$

(b) $$\int \frac{4x-7}{2x^2+1}\,dx = \int\left(\frac{4x}{2x^2+1} - \frac{7}{2x^2+1}\right)dx$$

$$= \int\left(\frac{4x}{2x^2+1} - \frac{7}{2}\cdot\frac{1}{x^2+\frac{1}{2}}\right)dx$$

$$= \ln(2x^2+1) - \frac{7}{2}\cdot\frac{1}{\sqrt{\frac{1}{2}}}\tan^{-1}\frac{x}{\sqrt{\frac{1}{2}}} + c$$

$$= \ln(2x^2+1) - \frac{7}{\sqrt{2}}\tan^{-1}(x\sqrt{2}) + c. \qquad (2)$$

(c)(i) $e^{-t} + e^{-2t} + e^{-3t} \ldots = \dfrac{e^{-t}}{1 - e^{-t}}$ as $|e^{-t}| < 1$ for $t > 0$. $\qquad (1)$

$$= \frac{1}{e^t - 1}$$

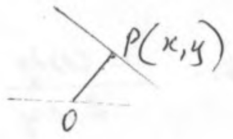
(ii) Now $\dfrac{d}{dt}\left[e^{-t} + e^{-2t} + e^{-3t} + \ldots\right] = \dfrac{d}{dt}\left(e^t - 1\right)^{-1}$

$$-e^{-t} - 2e^{-2t} - 3e^{-3t} + \ldots = -\left(e^t - 1\right)^{-2}\cdot e^t$$

$$\therefore\ e^{-t} + 2e^{-2t} + 3e^{-3t} \ldots = \frac{e^t}{(e^t-1)^2} \qquad (1)$$

5

(d) $y=\sqrt{r^2-x^2}$



Gradient Tangent $m_1 = \frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}}$ (1)

Gradient OP $= \frac{y-0}{x-0}$

$= \frac{y}{x}$

$m_2 = \frac{\sqrt{r^2-x^2}}{x}$

∴ $m_1 \times m_2 = \frac{-x}{\sqrt{r^2-x^2}} \cdot \frac{\sqrt{r^2-x^2}}{x}$

$= -1$ (1)

∴ Tangent ⊥ radius.

5 $(a+b)^n \equiv \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$

∴ $T_{r+1} = \binom{11}{r}(4x)^{11-r} 5^r$

$T_r = \binom{11}{r-1}(4x)^{12-r} 5^{r-1}$

$\frac{T_{r+1}}{T_r} = \frac{11!}{r!(11-r)!} \cdot \frac{(r-1)!(12-r)!}{11!} \cdot \frac{4^{11-r}}{4^{12-r}} \cdot \frac{5^r}{5^{r-1}} \cdot \frac{x^{11-r}}{x^{12-r}}$ (1)

$= \frac{12-r}{r} \cdot \frac{5}{4} \cdot \frac{1}{x}$

For largest coefficient $\frac{5(12-r)}{4r} \geq 1$ (1)

$60-5r \geq 4r$

$9r \geq 60$

$r \geq 7$ $T_8 < T_7$

∴ Largest coefficient $\binom{11}{6} 4^5 5^6$ (1)

6

(b) (i) No motion does not oscillate. 1

(ii) $\frac{d}{dx}\left(\frac{v^2}{2}\right) = \frac{d}{dv}\left(\frac{v^2}{2}\right) \cdot \frac{d}{dx} v$

$= \frac{2v}{2} \cdot \frac{dv}{dx}$

$= v \frac{dv}{dx}$

$= \frac{dx}{dt} \cdot \frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \ddot{x}$ 2

(iii)

$\ddot{x} = -625x$

$\frac{d}{dx}\left(\frac{v^2}{2}\right) = -625x$

$\frac{v^2}{2} = -625\frac{x^2}{2} + c$ But $v=0$ $x=-1$

∴ $0 = -\frac{625}{2} + c$

$c = +\frac{625}{2}$

∴ $\frac{v^2}{2} = \frac{625}{2} - 625\frac{x^2}{2}$

$v^2 = 625(1-x^2)$

$v = 25\sqrt{1-x^2}$ $v>0$. 2

(iv) At Surface $x=0$ ∴ $v = 25$ m/s 1

(v) $\frac{d^2x}{dt^2} = -g$

$\frac{d}{dx}\left(\frac{v^2}{2}\right) = -10$

(c) $\frac{v^2}{2} = -10x + C$ $\quad x=0 \quad v=25$

$\therefore C = \frac{25^2}{2}$

$\frac{v^2}{2} = -10x + \frac{625}{2}$

$v^2 = -20x + 625$

$x = \frac{625 - v^2}{20}$ (2)

(vi) Max height $v = 0$

$x = \frac{625}{20}$ (1)

$= 31.25\,m.$

6(a) Ways $= \binom{6}{2}\binom{8}{2} = 420$ (2)

(b) (i) Number words $\frac{6!}{3!} = 120$ (2)

(ii) C △ C △ C △ $\quad$ △ C △ C △ C

C △ △ C △ C

C △ C △ △ C (2)

Total $= 4 \times 3!$

$= 24$

(iii) CCC Y L I $\quad$ or $\quad$ I CCC YL

Ways I $= 2!$

Ways C $= 1$

Total $= 2 \times 3!$

I end $= 12$

Ways I $= 2!$

Ways $[\text{CCCYL}] = \frac{5!}{3!} = 20$

Total $= 2 \times 20$

$= 40$

$$\text{Probability}\left(\begin{array}{c}\text{If end is I} \\ \text{Cs together}\end{array}\right) = \frac{12}{40} = \frac{3}{10}$$ (2)

(b) Step 1 $\quad n = 1$

LHS $= \sin q \qquad$ RHS $= \frac{1 - \cos 2q}{2 \sin q}$

$= \frac{1 - (1 - 2\sin^2 q)}{2 \sin q}$

$= \frac{2\sin^2 q}{2 \sin q}$

$= \sin q$

$=$ LHS (1)

Step 2 $\quad \therefore$ True $n = 1$.

Assume true $n = k$ $\quad \sin q + \sin 3q + \ldots \quad \sin(2k-1)q = \frac{1 - \cos 2kq}{2 \sin q}$

To prove true $n = k+1$ $\quad \sin q + \sin 3q + \ldots \quad \sin(2k+1)q = \frac{1 - \cos 2(k+1)q}{2 \sin q}$

LHS $= \sin q + \sin 3q + \ldots + \sin(2k-1)q + \sin(2k+1)q$

$= \frac{1 - \cos 2kq}{2 \sin q} + \sin(2k+1)q$ (By assumption)

$= \frac{1 - \cos 2kq + 2\sin q \sin(2k+1)q}{2 \sin q}$ (1)

$= \frac{1 - \cos[(2k+1)q - q] + 2\sin q \sin(2k+1)q}{2 \sin q}$

$= \frac{1 - \{\cos(2k+1)q \cos q + \sin(2k+1)q \sin q\} + 2\sin q \sin(2k+1)q}{2 \sin q}$ (1)

$= \frac{1 - \{\cos(2k+1)q \cos q - \sin q \sin(2k+1)q\}}{2 \sin q}$

$= \frac{1 - \cos[(2k+1)q + q]}{2 \sin q}$ (1)

$= \frac{1 - \cos 2(k+1)q}{2 \sin q}$

$\therefore$ If statement true $n = k$ it is also true $n = k+1$.

Since true for $n = 1$ it is also true for $n = 1+1 = 2$, $n = 2+1 = 3$ and so on

i) [10] $\frac{dP}{d\theta} = r\left[2 - 2\cos\left(\frac{\pi}{3}-\theta\right)\right]$

$\frac{d^2P}{d\theta^2} = 2r\sin\left(\frac{\pi}{3}-\theta\right)$

[11]

For maximum perimeter $\frac{dP}{d\theta} = 0$

$r\left[2 - 2\cos\left(\frac{\pi}{3}-\theta\right)\right] = 0$

$r \neq 0 \qquad \cos\left(\frac{\pi}{3}-\theta\right) = 1$

$\frac{\pi}{3} - \theta = 0$ for $0 < \theta < \frac{\pi}{2}$ /

$\therefore \theta = \frac{\pi}{3}$

For nature test $\frac{d^2P}{d\theta^2}$ for concavity

at $\theta = \frac{\pi}{3}$ $\quad \frac{d^2P}{d\theta^2} = r \times 0$

$= 0$

Test gradients:

θ	1	$\frac{\pi}{3}$	1.1
$\frac{dP}{d\theta}$	$2\times10^{-3}r$	0	$2\times10^{-3}r$

/

gradients same sign / — /

$\therefore$ Inflexion point at $\theta = \frac{\pi}{3}$ and monotonic increasing continuous for $0 < \theta < \frac{\pi}{2}$

$\therefore$ Maximum perimeter at end points of domain /

ie $\theta = \frac{\pi}{2}$.